Bayesian Statistical Inference

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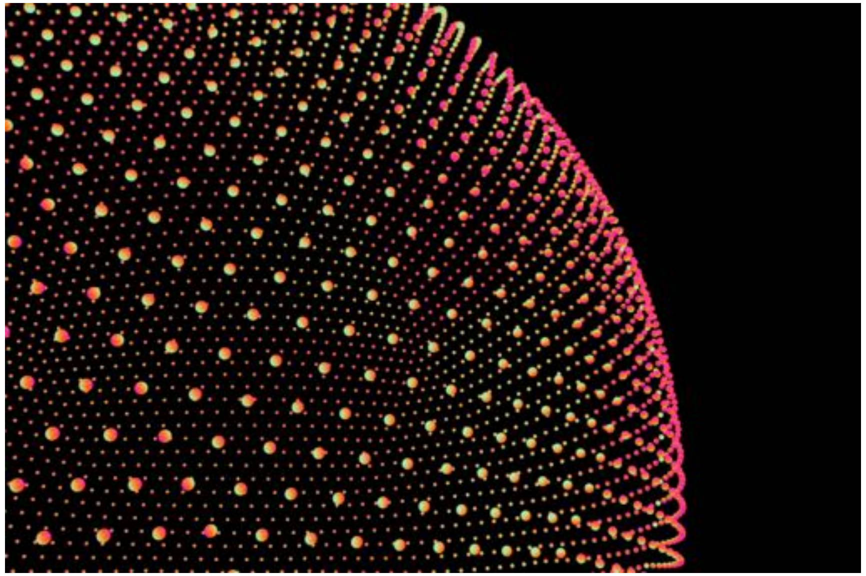
Contents

Statistical inference

”Statistical inference is the process of drawing conclusions about an ***underlying population*** *based on a* ***sample*** *or subset of the data.”*

Thomas D. Gauthier, Mark E. Hawley (2015). Introduction to Environmental Forensics (Third Edition).

Statistical inference = Translating data into understanding

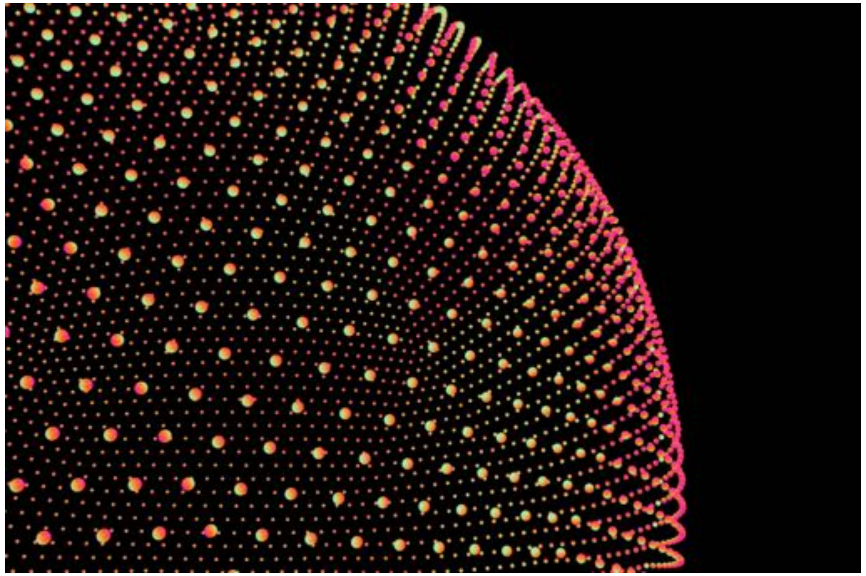


Data

Underlying population

Thomas D. Gauthier, Mark E. Hawley (2015). Introduction to Environmental Forensics (Third Edition).; PowerPoint image bank.

Statistical inference = Translating data into understanding

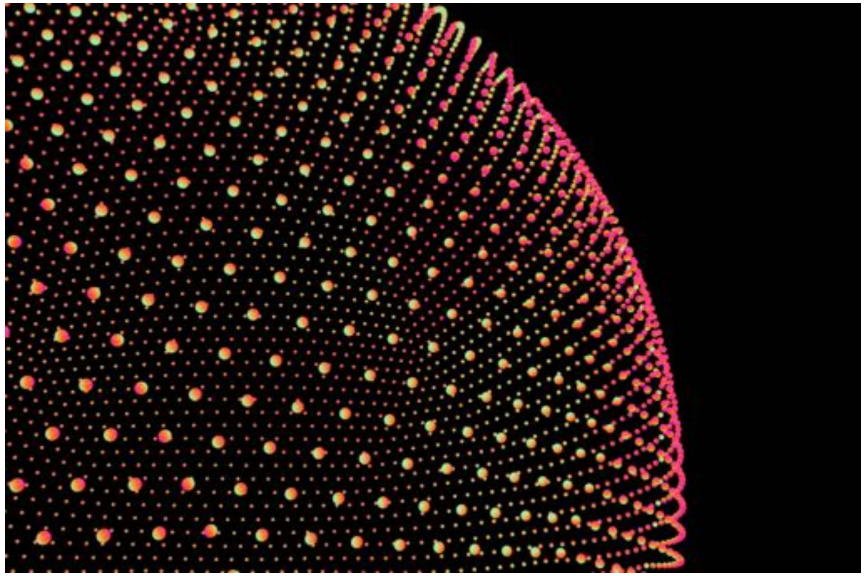


Example: How much people love statistics?

- Subset: not practical to ask everyone
- Sample bias: poll at stats dept. – generalize to others?
- Measuring variation: bad day?

Holopainen, M. & Pulkkinen P. (2012). *Tilastolliset menetelmät*. Sanoma Pro Oy.; PowerPoint image bank.

Statistical inference



$\neq$

→ Dealing with **uncertainty**:

Gap

The heart and soul of statistical inference

McElreath, R. (2018). Statistical rethinking: A Bayesian course with examples in R and Stan. Chapman and Hall/CRC.; PowerPoint image bank.

Describing uncertainty: Probability

0 ——— Likelihood of an event ——— 1

$$P(\text{green}) = \frac{1}{5} = 0.2$$

Holopainen, M. & Pulkkinen P. (2012). *Tilastolliset menetelmät*. Sanoma Pro Oy.; PowerPoint image bank.

Describing uncertainty: Probability

0 — Likelihood of an event — 1

$$P(\text{green}) = \frac{1}{5} = 0.2$$

Rarely that simple in research!
But a key concept that can be applied in various ways

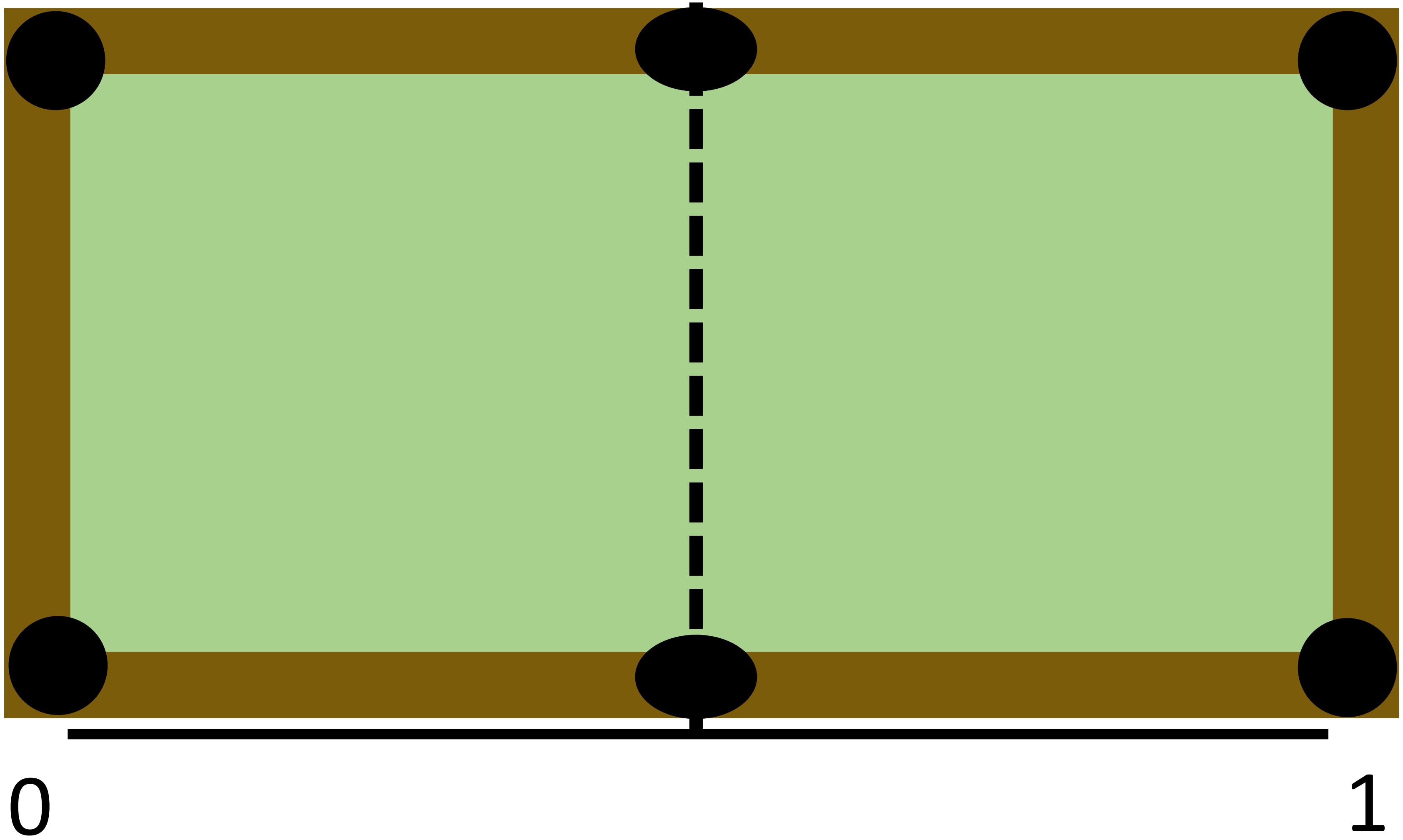
Holopainen, M. & Pulkkinen P. (2012). *Tilastolliset menetelmät*. Sanoma Pro Oy.; PowerPoint image bank.

Two approaches to statistical inference

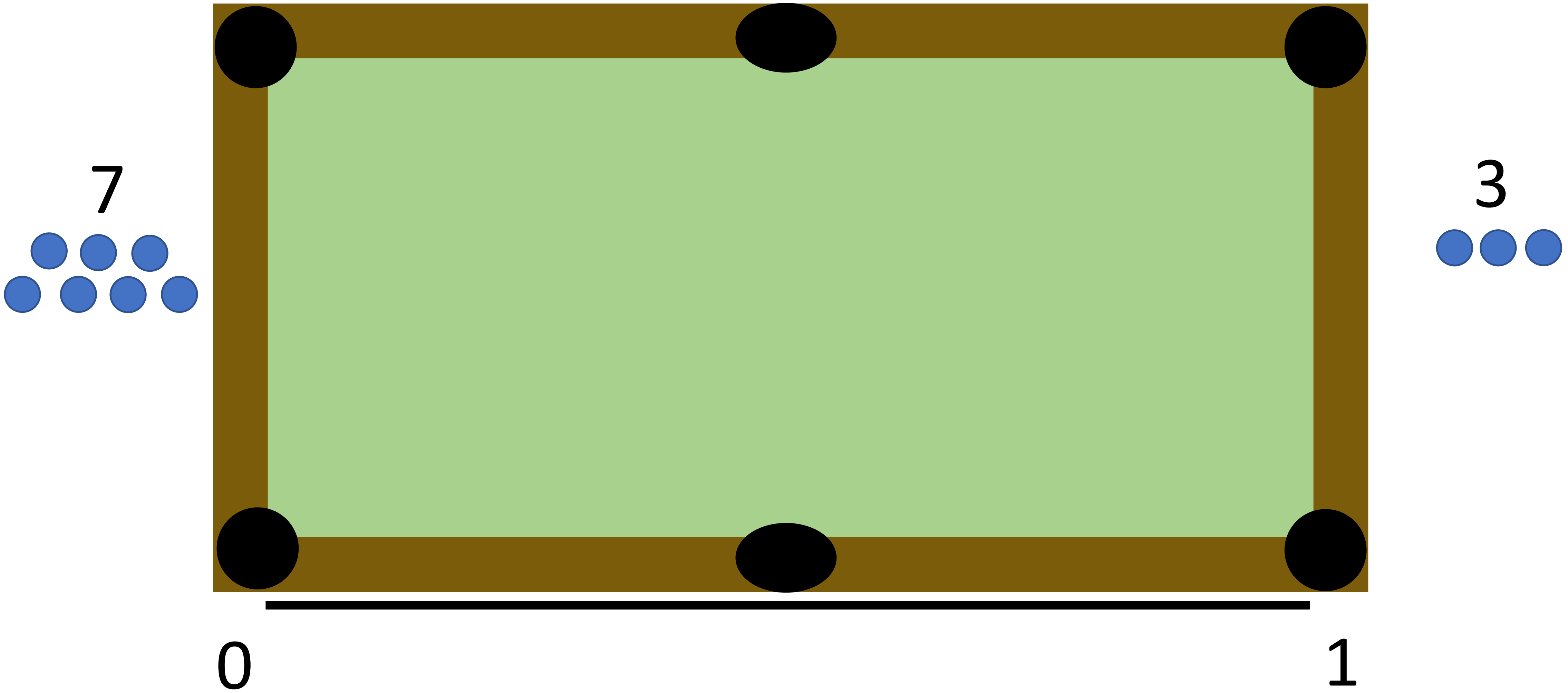
1. Bayesian
2. Frequentist

Gelman, A., Carlin, J. B., Stern, H. S., & Rubin, D. B. (1995-2020). Bayesian data analysis. Chapman and Hall/CRC.; PowerPoint image bank.

Thomas Bayes



Gower, B. (1997). Scientific method. Routledge.



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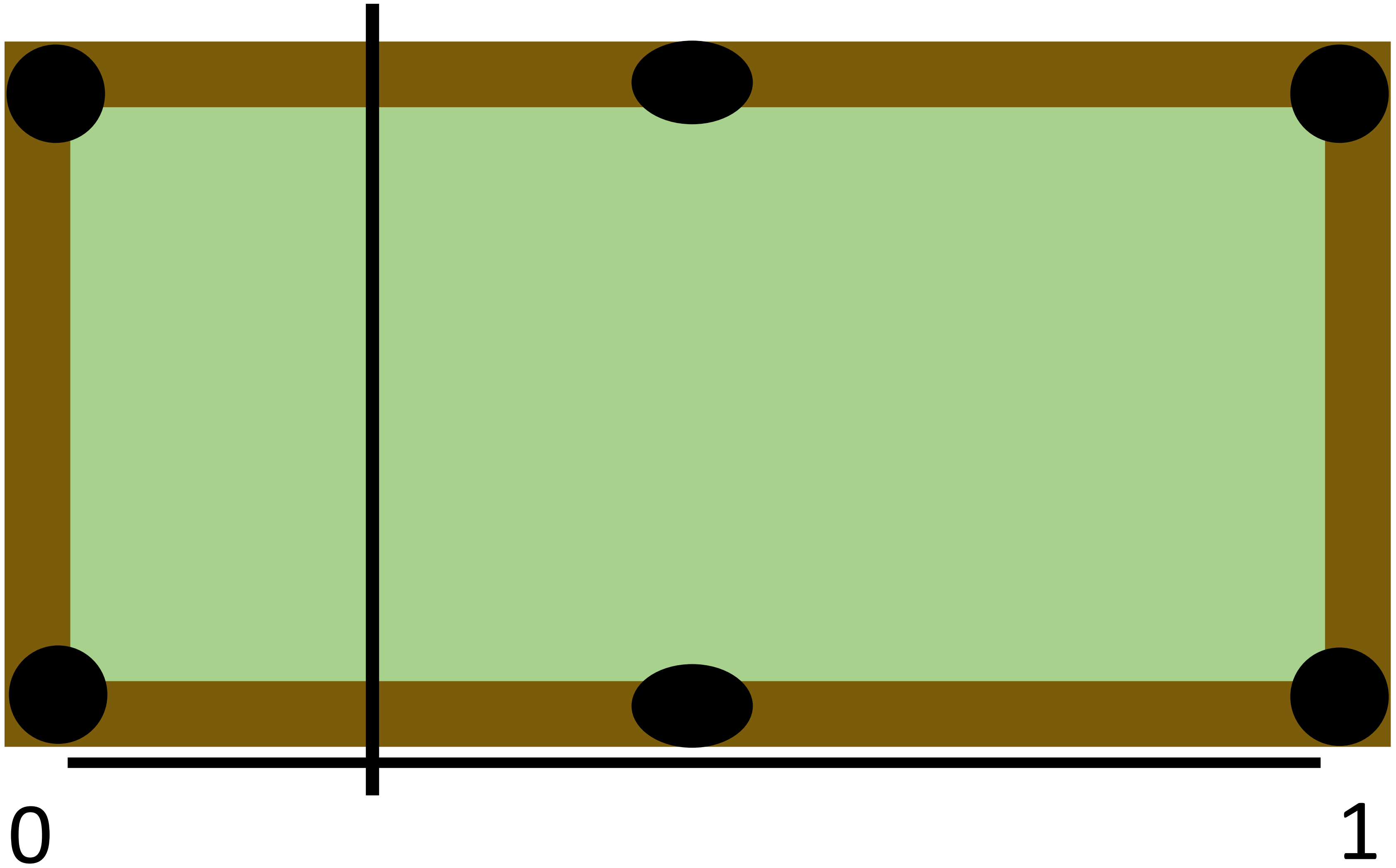

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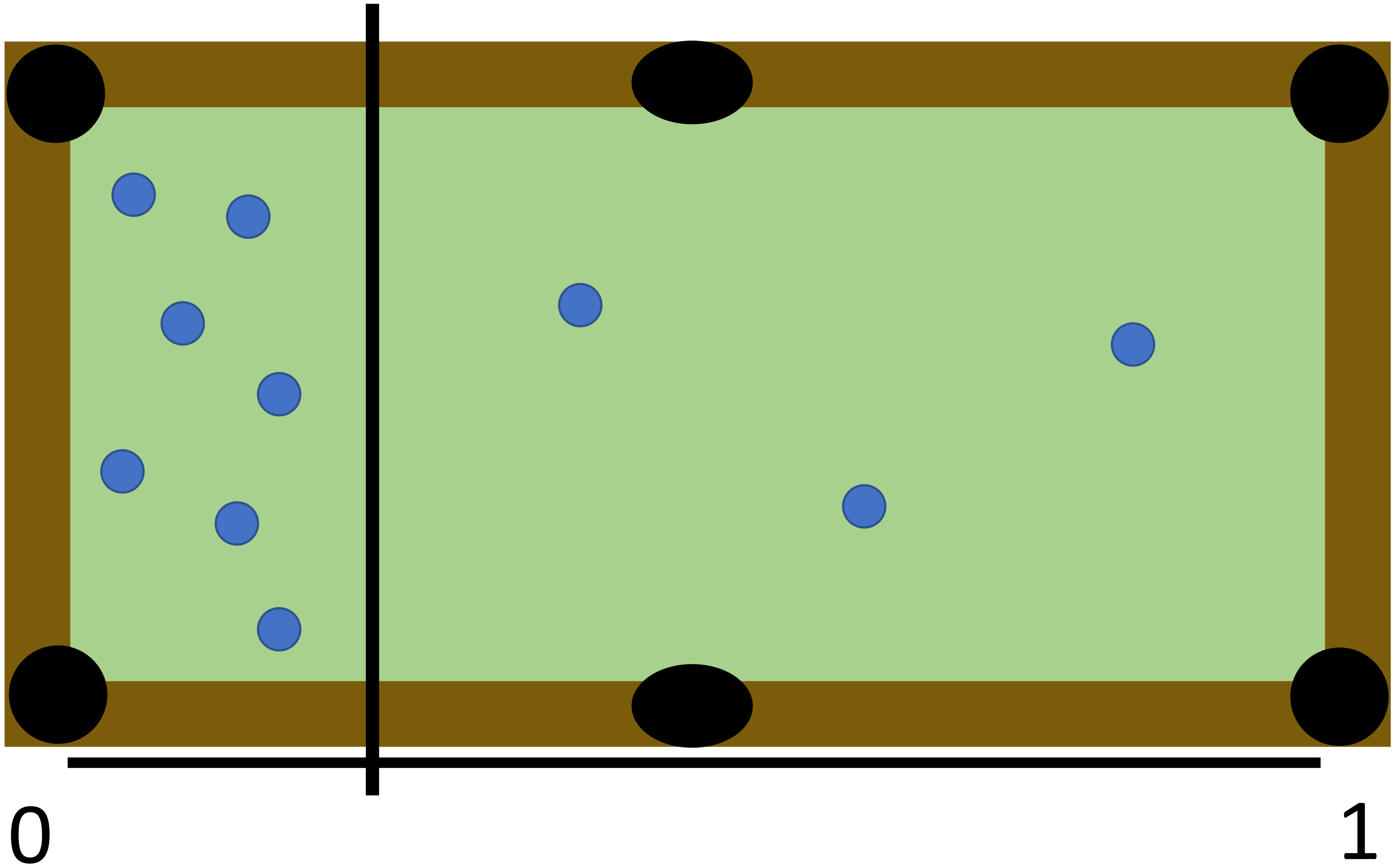

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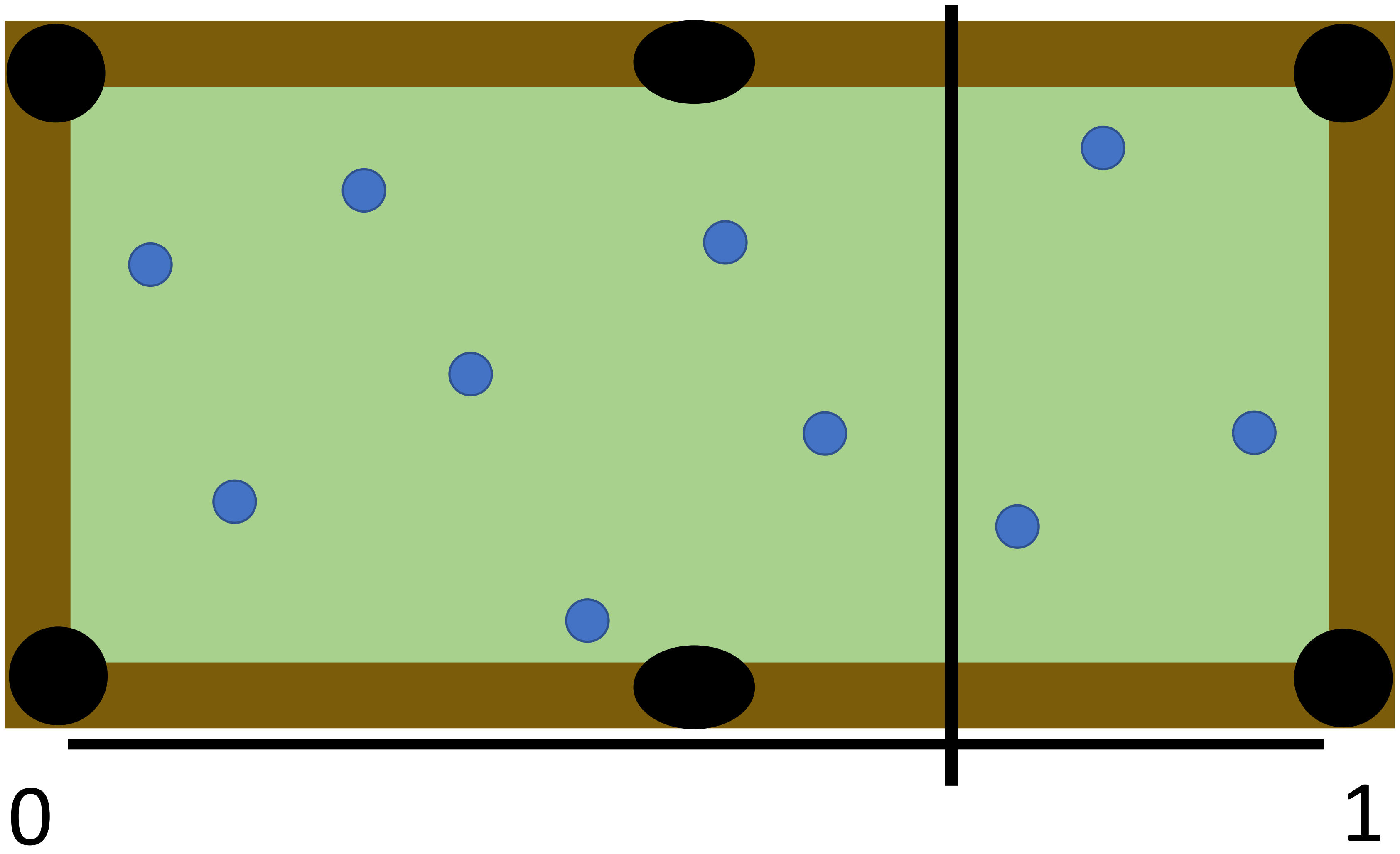
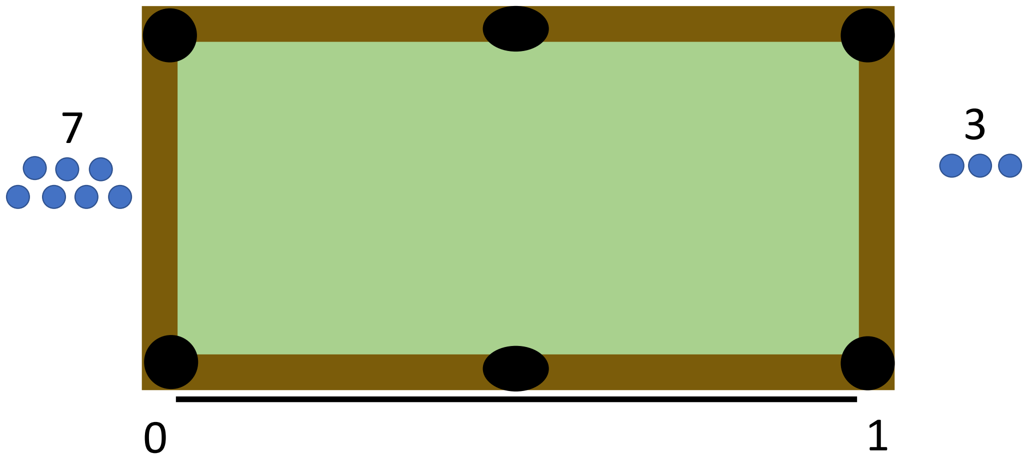


Gower, B. (1997). Scientific method. Routledge.

Where is the line?



1) What do we know prior to data?

- There is a line, somewhere from 0 to 1

2) What do we observe?

- 7 on the left
- 3 on the right

Both the **prior** knowledge and the **data** (observations) affect our **posterior** understanding

Gelman, A., Carlin, J. B., Stern, H. S., & Rubin, D. B. (1995-2020). Bayesian data analysis. Chapman and Hall/CRC.

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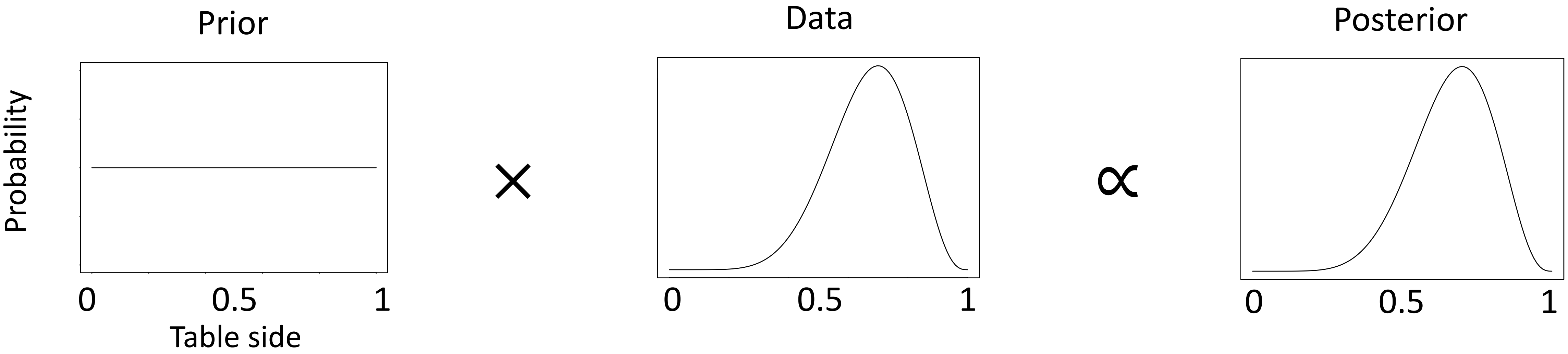
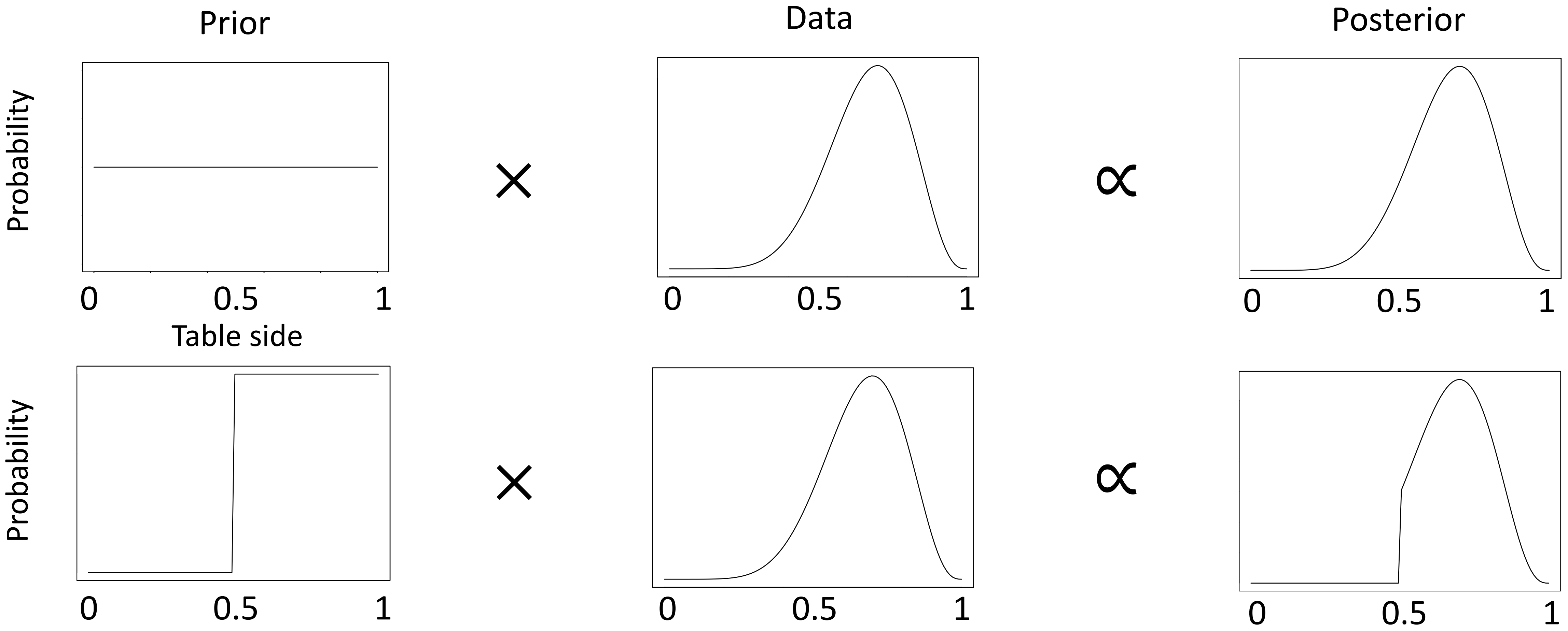


McElreath, R. (2018).; R Core Team (2020). R: A language and environment for statistical computing.

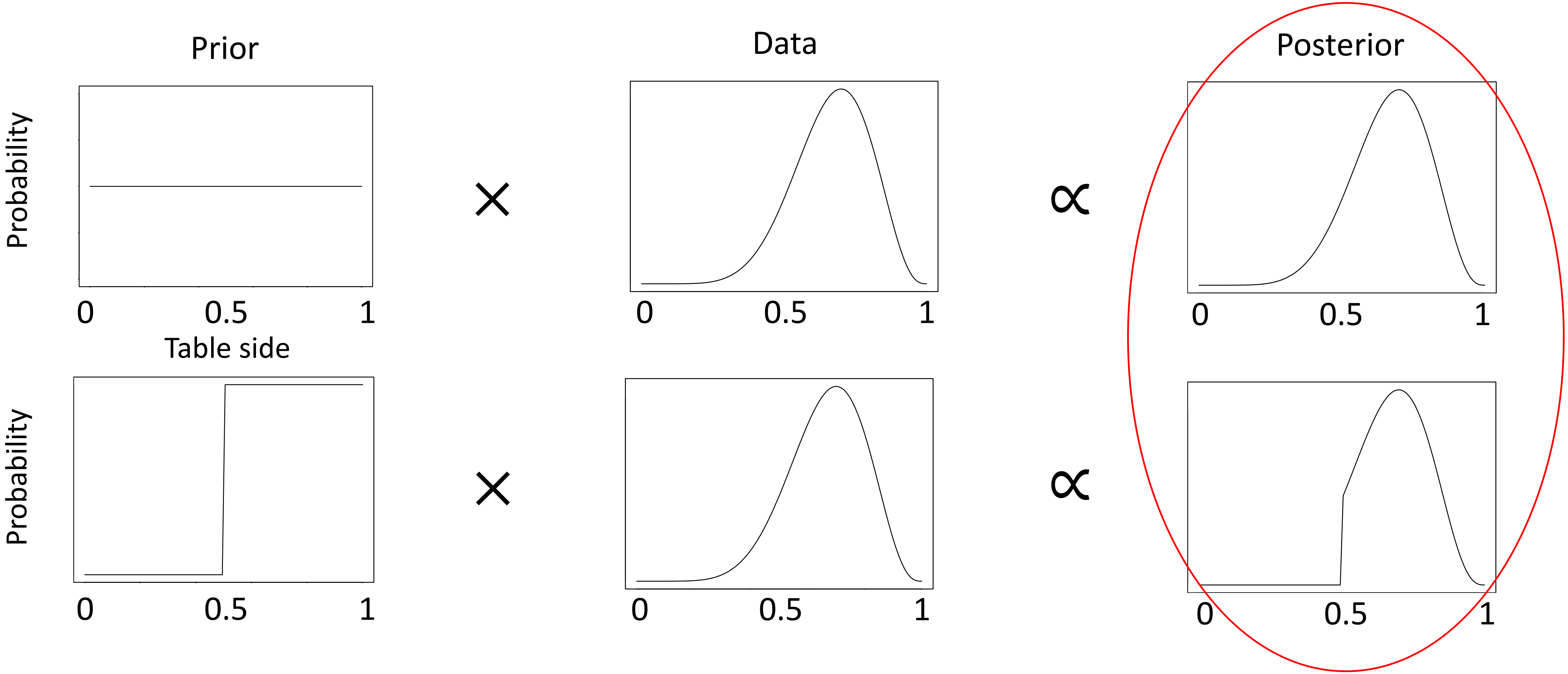
Where is the line?



McElreath, R. (2018).; R Core Team (2020). R: A language and environment for statistical computing.

Where is the line?

Can be an iterative process:
If we got more data,
we could use the posterior as the new pior



McElreath, R. (2018).; R Core Team (2020). R: A language and environment for statistical computing.

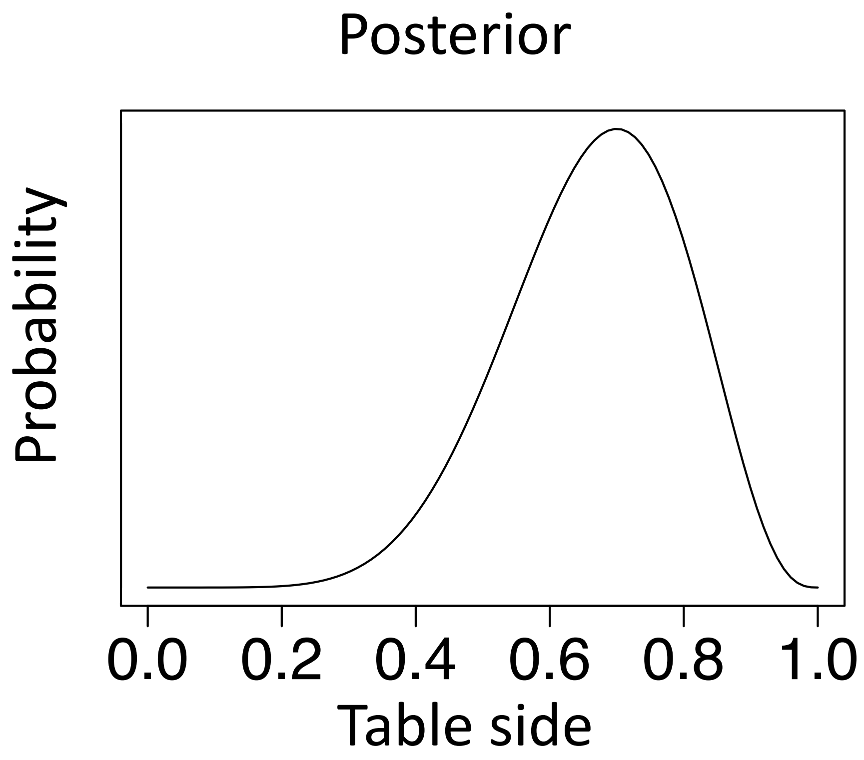
The result is a probability distribution for the parameter

Parameter= the unknown ground truth we try to solve

Probability for each possible location of the line
But the line is somewhere, we just don't know it

(50% probability one has a virus, is pregnant…)
What does it mean to be 50% likely pregnant?
One is or not

Subjective probability:
- Give parameter probabilities
- Get posterior probabilities back (updated with data)



Where is the probability?

Bayesian accepts **subjective**:
Given the data, what is the probability distribution of the parameter?

Data

Sample
Observable
Known
(everybody accepts probability here)

Parameter

Population
Unknown
Interest
’State of world’
Truth

McElreath, R. (2018).; Holopainen, M. & Pulkkinen P. (2012). *Tilastolliset menetelmät*. Sanoma Pro Oy.

Where is the probability?

Data

Sample
Observable
Known
(everybody accepts probability here)

Bayesian accepts **subjective**:
Given the data, what is the probability distribution of the parameter?

Frequentist does not:
Assuming the parameter (H0), how probable is the data?

Parameter

Population
Unknown
Interest
'State of world'
Truth

McElreath, R. (2018).; Holopainen, M. & Pulkkinen P. (2012). *Tilastolliset menetelmät*. Sanoma Pro Oy.

Example: Laugh

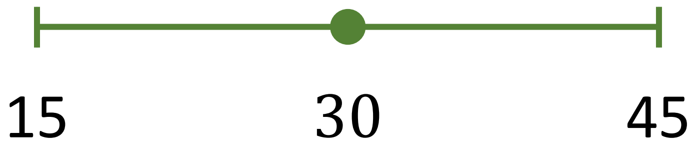
- How much people laugh?

- Sample n= 100 people
- Laughtermeter: Measure how long a subject laughs during a day

- Parameter: true mean laughing time in the population

Example: Laugh

Bayesian 95% credible interval



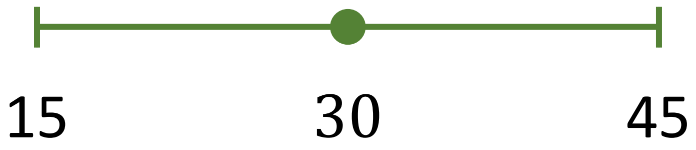
95% probability that people laugh 15-45 min a day
Most likely 30 min

McElreath, R. (2018).; Data: mock-up!

θ = true parameter value

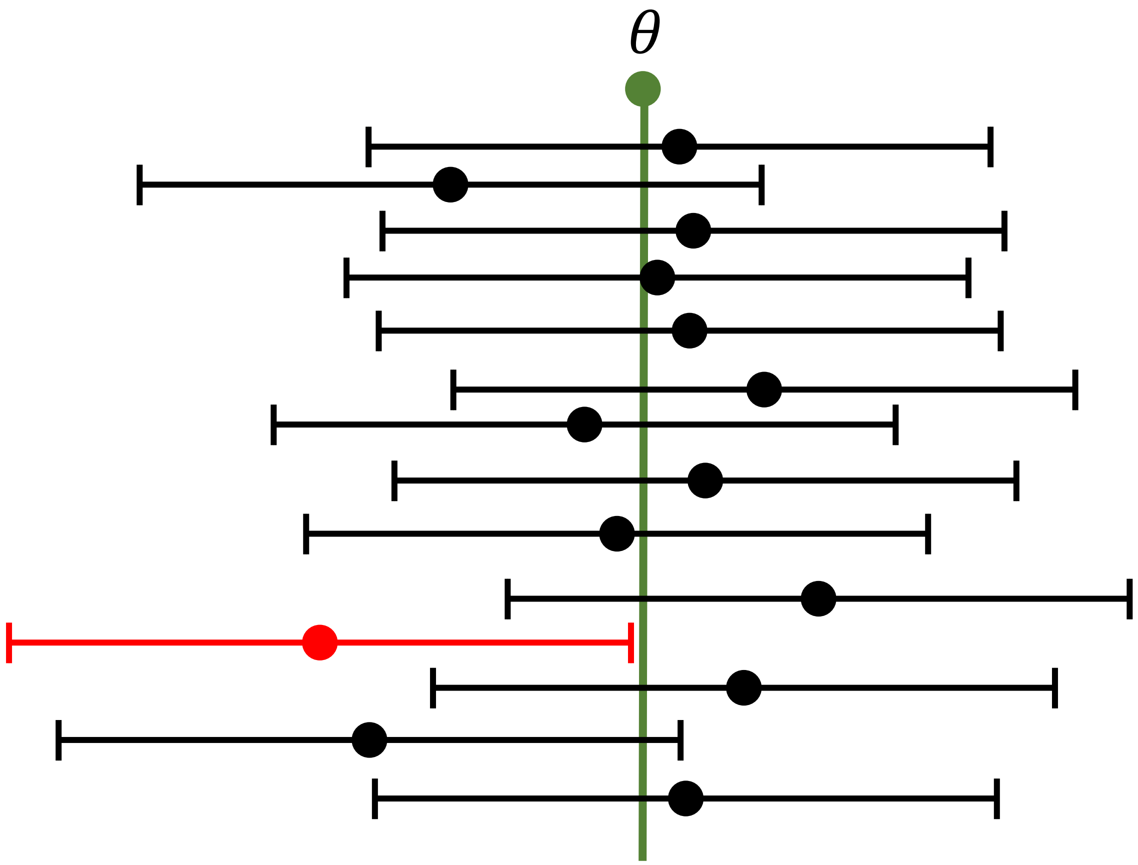
Example: Laugh

Bayesian 95% credible interval



95% probability that people laugh 15-45 min a day
Most likely 30 min

Frequentist 95% confidence interval



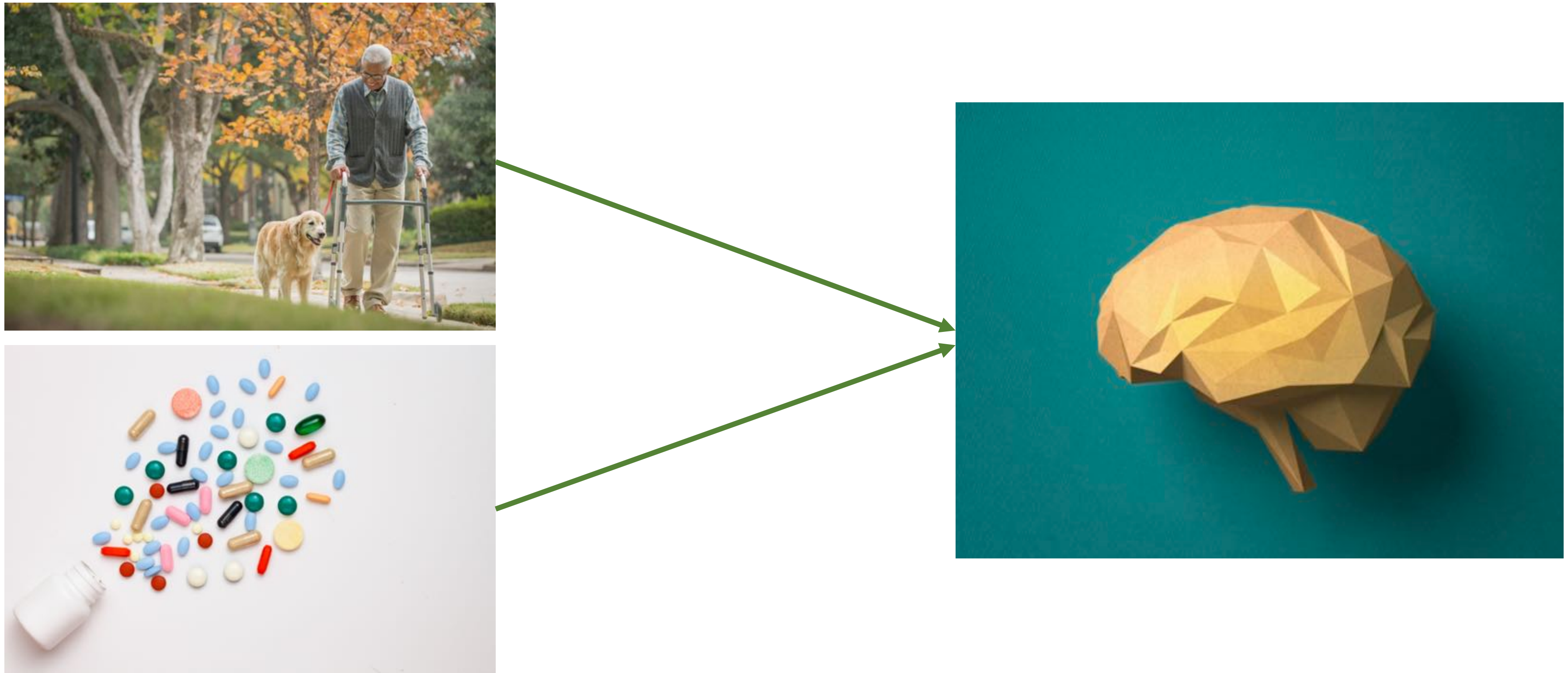
When we take many many samples and calculate confidence interval for each, 95% of the times the true mean laughing time is within the interval

McElreath, R. (2018).; Cumming, G. (2013). Intro Statistics 9 Dance of the p Values.; Holopainen, M. & Pulkkinen P. (2012).; Data: mock-up!

Contents

Usual interest

- How some variable is related to another



McElreath, R. (2018).; PowerPoint image bank.

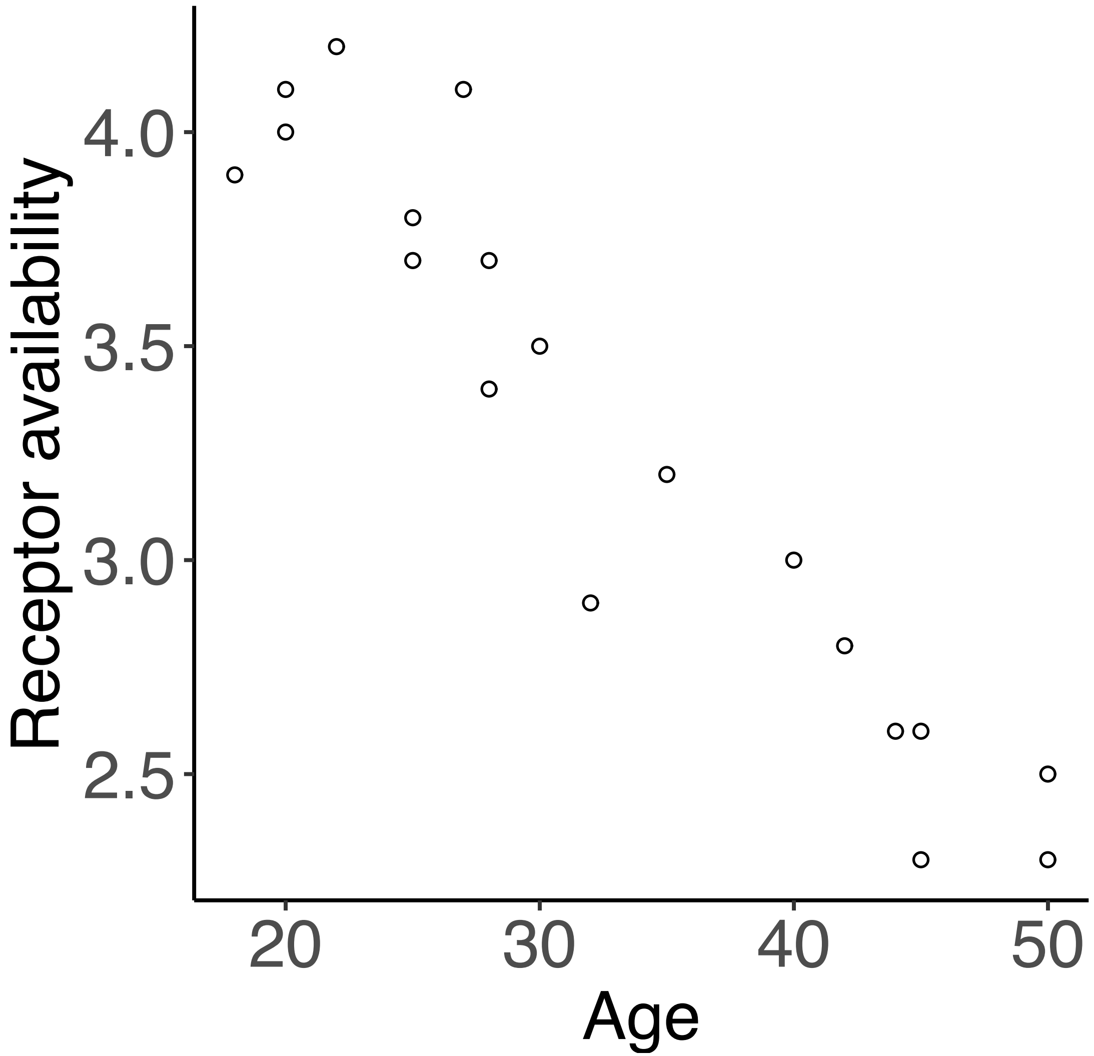
Example

How is **age** associated with (dopamine) **receptor availability** in a region of interest (ROI)?

Receptor availability (binding potential = bp)

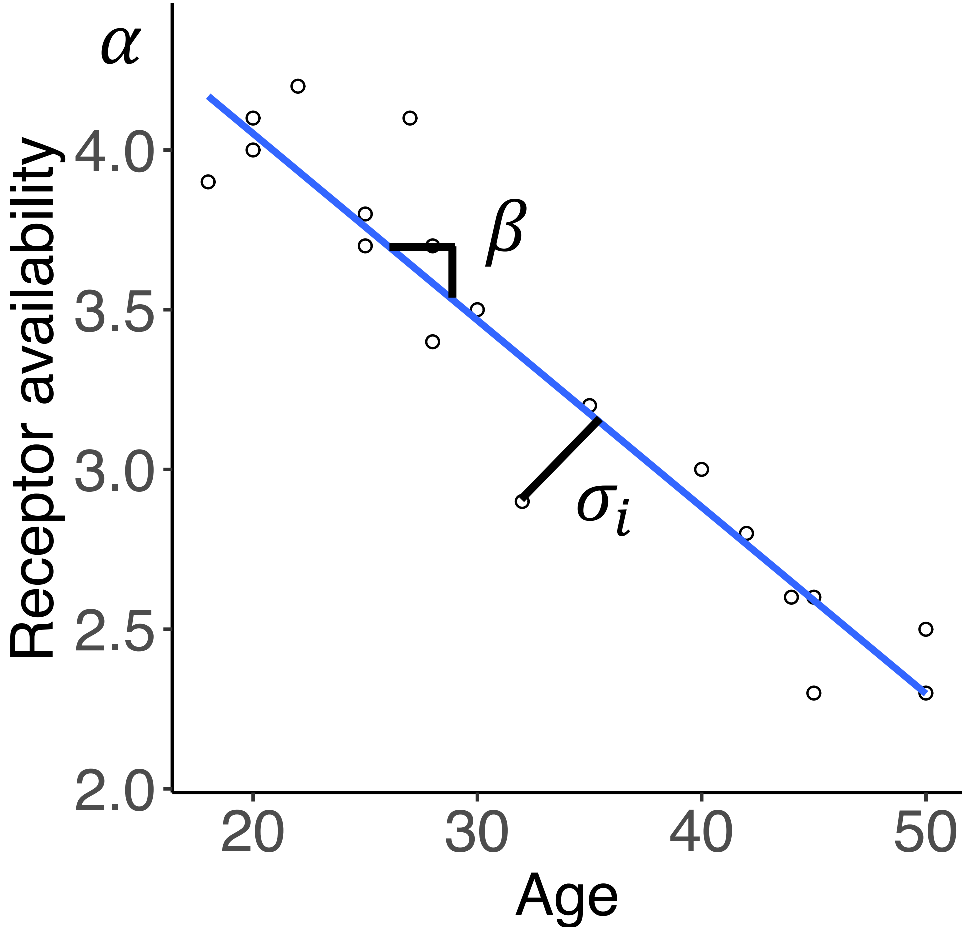
- Always positive value acquired from a PET image (1 estimate/ ROI for each subject)
- The density (and affinity) of available (unoccupied) dopamine receptors in a specific region → information about dopamine function
- $\text{Normal}(\mu, \sigma)$

Innis et al. (2007). Consensus nomenclature for in vivo imaging of reversibly binding radioligands. Journal of Cerebral Blood Flow & Metabolism, 27(9), 1533-1539.



- n= 20
- Age
- Receptor availability (bp) from PET
- 1 brain region

Linear regression: linear relation

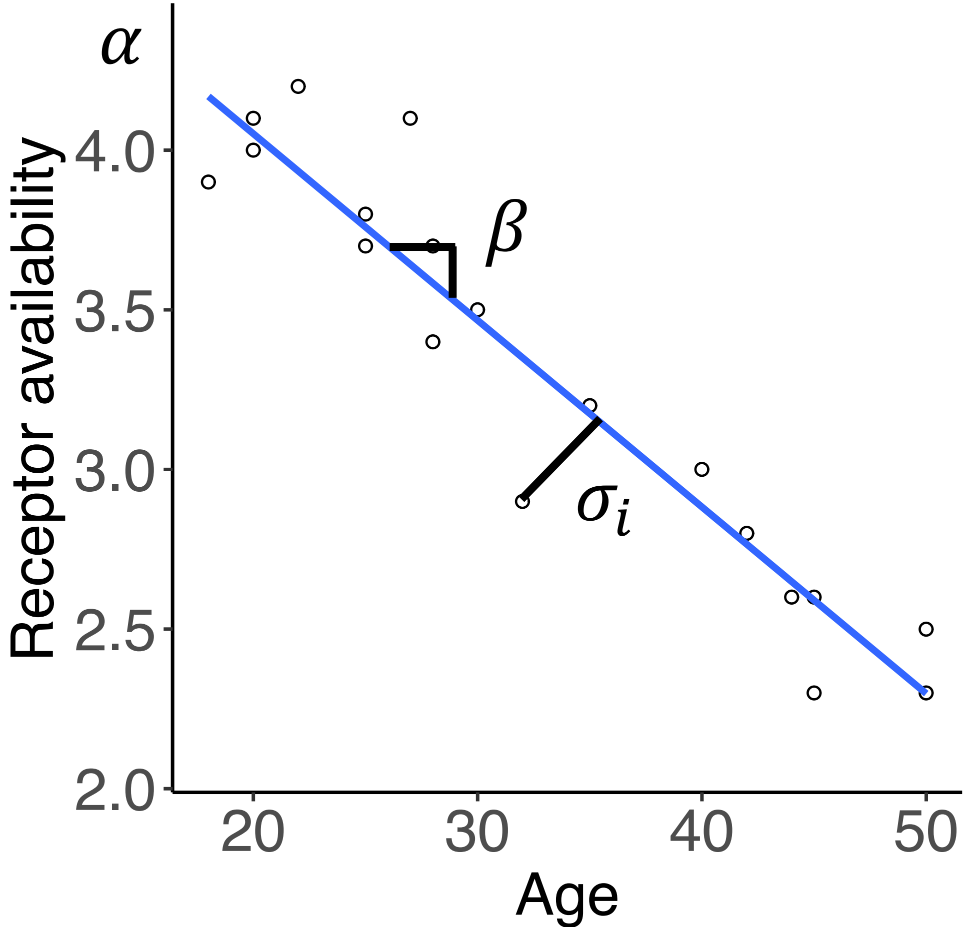


Parameters:
α = expected availability, when age = 0

β = regression coefficient (slope), steepness of the line, the change in availability when age increases 1 unit

σ_i = distance between an observation and regression line

Linear regression: linear relation



Parameters:
α = expected availability, when age = 0

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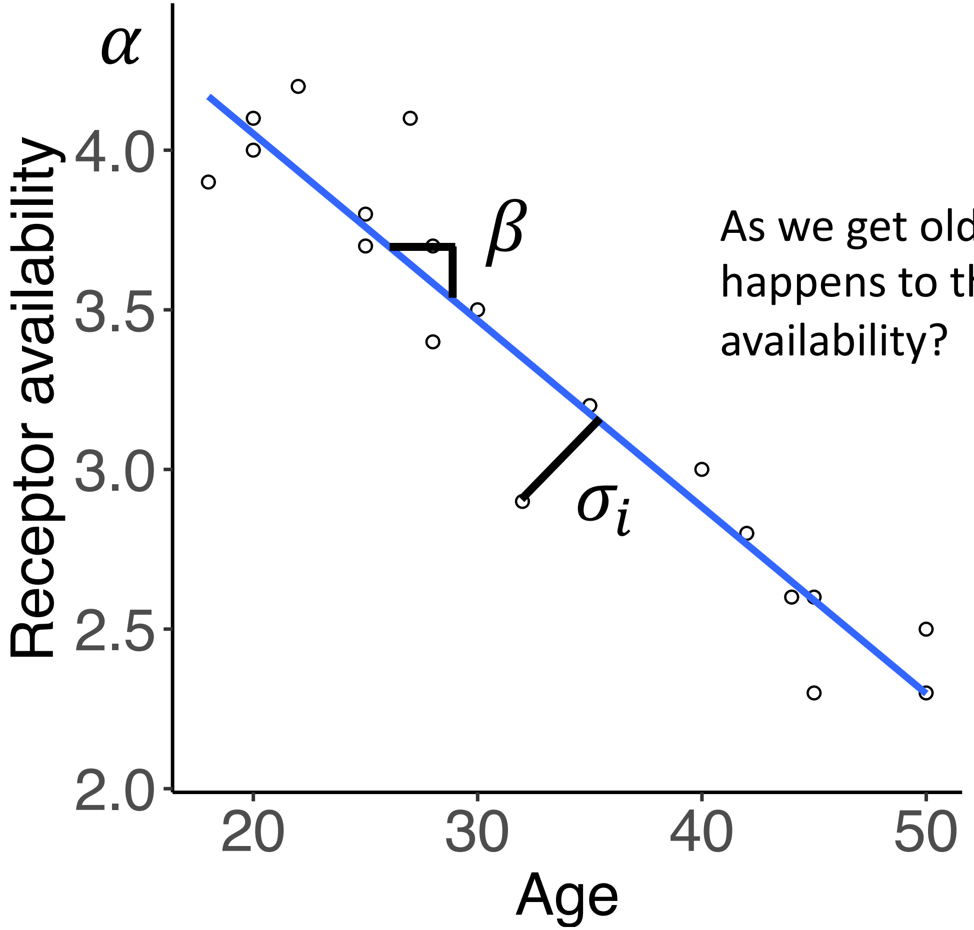
σ_i = distance between an observation and regression line

Linear model:
$\mu = \alpha + \beta \times \text{age}$

McElreath, R. (2018).; R Core Team (2020).; Data: mock-up!

Linear regression: linear relation



As we get older, what happens to the receptor availability?

Parameters:
α = expected availability, when age = 0

β = regression coefficient (slope), steepness of the line, the change in availability when age increases 1 unit

σ_i = distance between an observation and regression line

Linear model:
$\mu = \alpha + \beta \times \text{age}$

Let’s solve β the Bayesian way

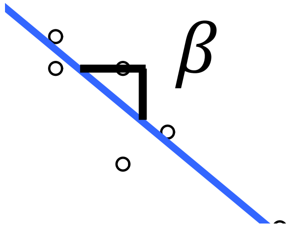
(= what happens to the receptor availability when we age)

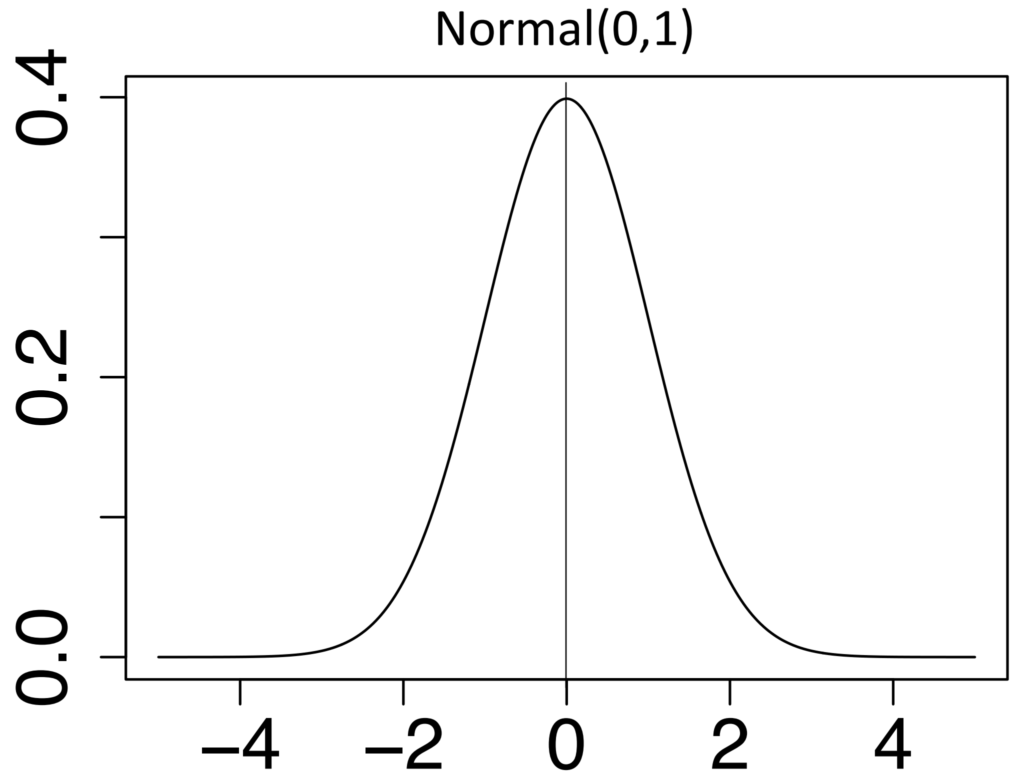
Tools

- brms-package in R

R Core Team (2020). Buerkner P-C, Buerkner MP-C (2016): Package ‘brms’.; Bürkner, P. C. (2017). brms: An R package for Bayesian multilevel models using Stan. Journal of statistical software, 80(1), 1-28.

Prior for each parameter



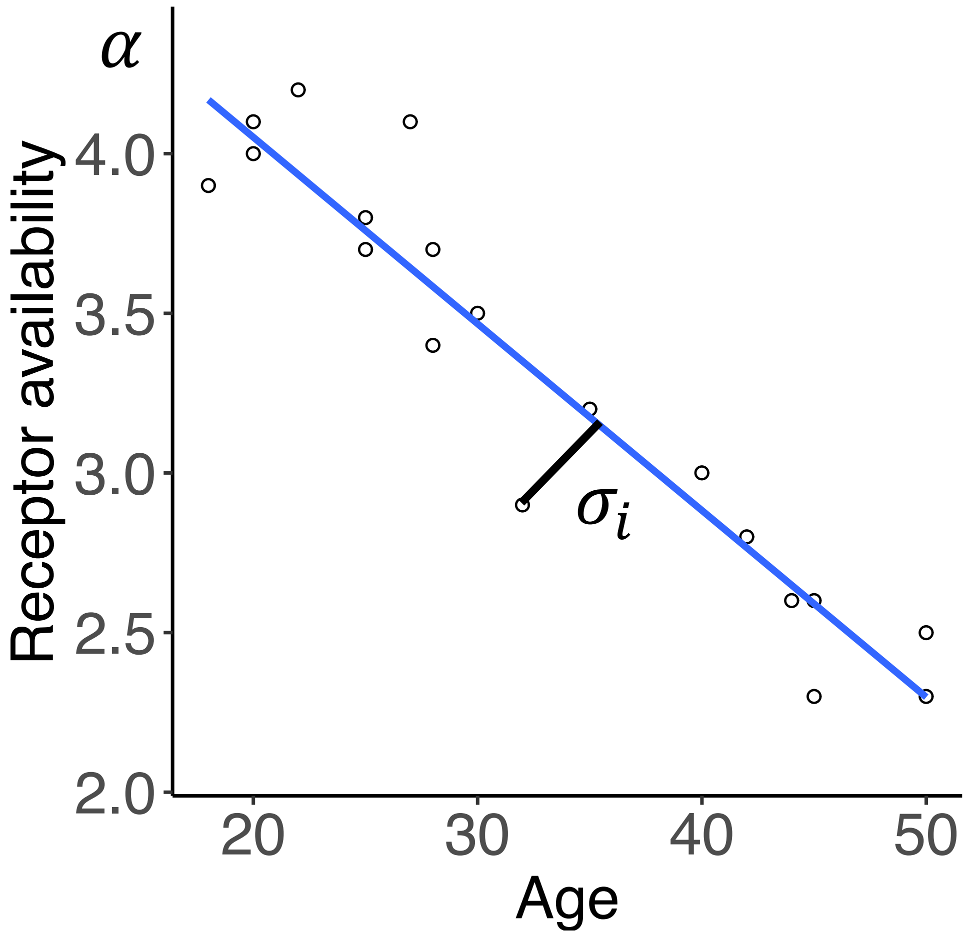


β = regression coefficient (slope)

- Mean is at ”zero-effect” (no association)
- The extremely high associations (steep line) are suggested less probable than weaker associations: does not push toward strong associations → if one found the evidence from data
- No values for β are excluded (no 0 probability given)
- Symmetrical: neither positive nor negative slope is given higher probability than the other

McElreath, R. (2018).; Buerkner P-C, Buerkner MP-C (2016).; Bürkner, P. C. (2017).; R Package Documentation. set_prior: Prior Definitions for 'brms' Models in brms: Bayesian Regression Models using 'Stan'; R Core Team (2020).

Prior for each parameter



Also α and σ are parameters of the model: need priors

For now brms package default priors (weakly informative)

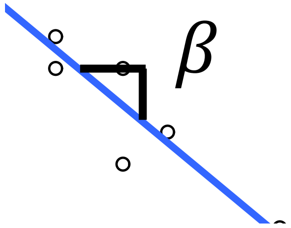
Good to be aware of all priors used and their fit in your data and population

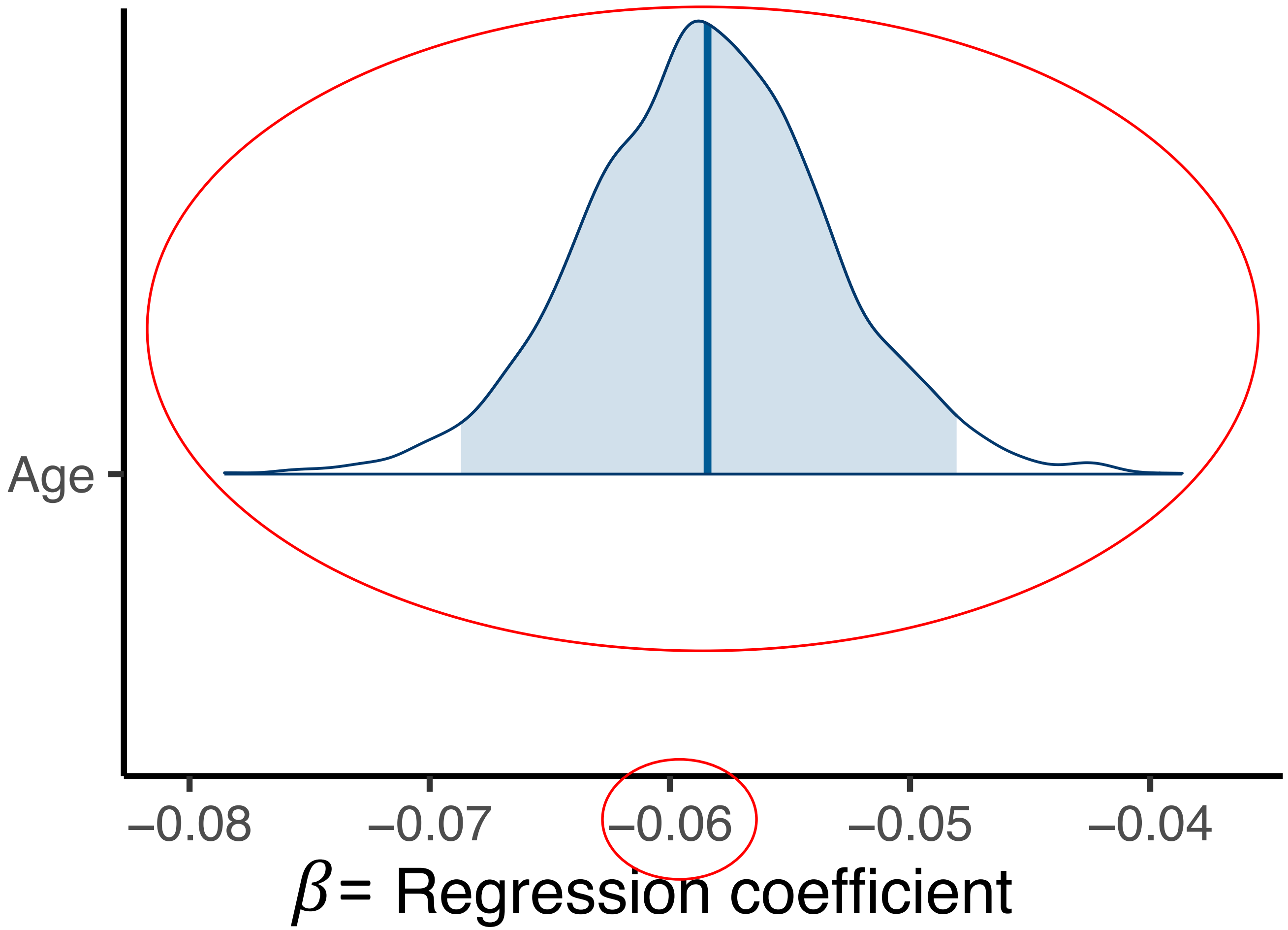
McElreath, R. (2018).; Buerkner P-C, Buerkner MP-C (2016); R Core Team (2020).

```
#### REQUIRED PACKAGES
library(rstan)
library(brms)

### STATISTICAL MODELING

prior <- c(set_prior("normal(0,1)", class = "b")) # prior for beta

model <- bp ~ age    # bp = receptor availability

fit <- brm(formula = model,
           data = data, # mock-up
           family = gaussian(), # receptor availability normally distributed
           prior = prior,
           warmup = 1000, iter = 2000, chains = 4, # sampling settings, see brms
           control = list(adapt_delta = 0.95)) # sampling settings, see brms
```

McElreath, R. (2018).; R Core Team (2020).; Buerkner P-C, Buerkner MP-C (2016).; Bürkner, P. C. (2017).; R Package Documentation. set_prior: Prior Definitions for 'brms' Models in brms: Bayesian Regression Models using 'Stan'; Bürkner, P. (2021). Rdocumentation. brms.

Result for β





Ageing 1 year,
The availability decreases 0.06 units
=
Ageing 10 years
The availability decreases 0.6 units

McElreath, R. (2018).; Bürkner, P. C. (2017).; R Core Team (2020).; Gabry J, Mahr T (2021). "bayesplot: Plotting for Bayesian Models." R package version 1.8.0.

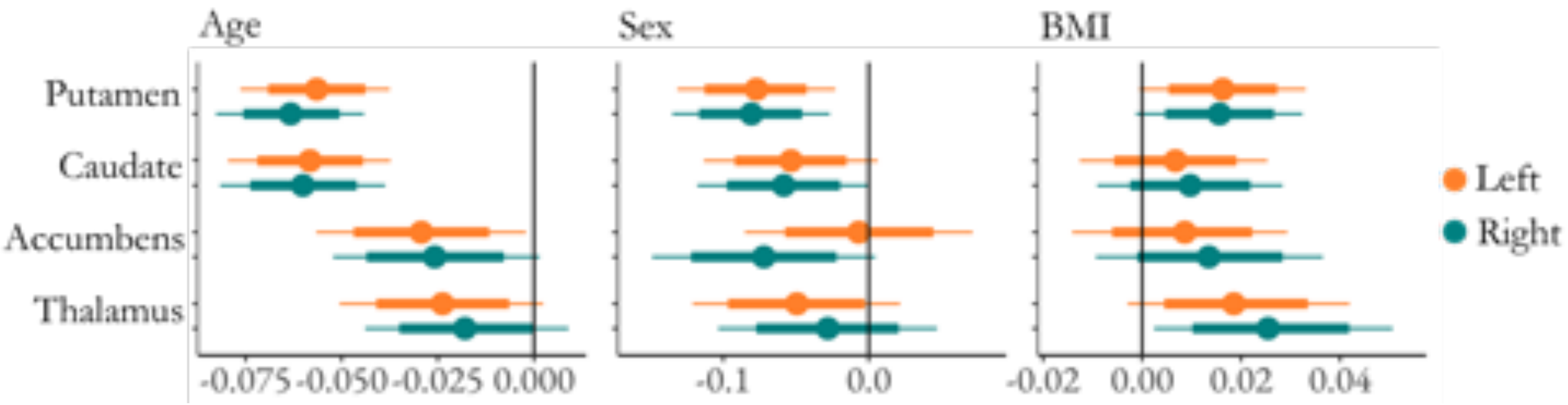
When?

- Simple enough setting
  - Computational methods (sampling from the posterior distribution)
  - Regional rather than voxel-level
  - Some predictors (e.g. age, sex…) but not too many

When?

Age, sex, BMI → dopamine receptor availability (preprint)

- Ageing 10 years and 5% decrease in the availability (putamen and caudate)
- Females had higher availability than males
- BMI: weak positive association with the availability
- Please see the preprint for more information about the scales to use (standardizing, log-transformation)



Bayesian methods not limited to linear regression

Malén T, Karjalainen T, Isojärvi J, Vehtari A, Bürkner P-C, Putkinen V, et al. (2021): Age and sex dependent variability of type 2 dopamine receptors in the human brain: A large-scale PET cohort. *bioRxiv*.2021.2008.2010.455776.; R Core Team (2020.; Gabry J, Mahr T (2021).; McElreath, R. (2018).

Why?

Interpretation intuitive

- Probabilities for parameter values (what we want)
  - Instead of having probability for the data in the light of only 1 assumed parameter value (null-hypothesis)
- One sample enough -> imaginary resampling not needed

To take home

- Prior and data form posterior

- If interested in how your prior affects the result, you can try different ones and compare your posteriors

- We have information about the world before we see the data and we can use it (as long as we accept the subjective probability)!

Further Learning Material

- Bayesian Data Analysis (Gelman, Carlin, Stern Rubin), Chapman Hall, 1995, 2003, 2013
  - http://www.stat.columbia.edu/~gelman/books

- McElreath, R. (2018). *Statistical rethinking: A Bayesian course with examples in R and Stan*. Chapman and Hall/CRC.
  - https://www.youtube.com/watch?v=4WVelCswXo4
  - Scripts and datasets

Acknowledgements

- **Tomi Karjalainen**, Data scientist, Polar Electro Oy, Turku PET Centre
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- **Vesa Putkinen**, Postdoctoral researcher, University of Turku
- **Birgitta Paranko**, Project researcher, University of Turku
- **Severi Santavirta**, MD, Doctoral candidate, University of Turku

References

- Thomas D. Gauthier, Mark E. Hawley (2015). Introduction to Environmental Forensics (Third Edition).
- McElreath, R. (2018). Statistical rethinking: A Bayesian course with examples in R and Stan. Chapman and Hall/CRC.
- Holopainen, M. & Pulkkinen P. (2012). *Tilastolliset menetelmät*. Sanoma Pro Oy.
- Gelman, A., Carlin, J. B., Stern, H. S., & Rubin, D. B. (1995-2020). Bayesian data analysis. Chapman and Hall/CRC.
- Gower, B. (1997). Scientific method. Routledge.
- Chick, S. E. (2006). Subjective probability and Bayesian methodology. *Handbooks in Operations Research and Management Science*, *13*, 225-257.
- Cumming, G. (2013). Intro Statistics 9 Dance of the p Values. Video. Retrieved from: https://www.youtube.com/watch?v=5OL1RqHrZQ8. Cited: 13.8.2021.
- Innis, R. B., Cunningham, V. J., Delforge, J., Fujita, M., Gjedde, A., Gunn, R. N., ... & Carson, R. E. (2007). Consensus nomenclature for in vivo imaging of reversibly binding radioligands. Journal of Cerebral Blood Flow & Metabolism, 27(9), 1533-1539.

R

- R Core Team (2020). R: A language and environment for statistical computing.
- Buerkner P-C, Buerkner MP-C (2016): Package ‘brms’.
- Bürkner, P. C. (2017). brms: An R package for Bayesian multilevel models using Stan. Journal of statistical software, 80(1), 1-28.
- R Package Documentation. set_prior: Prior Definitions for 'brms' Models in brms: Bayesian Regression Models using 'Stan’. Retrieved from: https://rdrr.io/cran/brms/man/set_prior.html. Cited: 16.8.2021.
- Bürkner, P. (2021). Rdocumentation. brms. Retrieved from: https://www.rdocumentation.org/packages/brms/versions/2.15.0. Cited: 17.8.2021.
- Gabry J, Mahr T (2021). “bayesplot: Plotting for Bayesian Models.” R package version 1.8.0, <URL:https://mc-stan.org/bayesplot/>.

Images

- PowerPoint image bank